Map from one-dimensional quantum field theory to quantum chaos on a two-dimensional torus

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Dynamics of a class of quantum field models on a one-dimensional (1D) lattice in a Heisenberg picture is mapped onto a class of quantum chaotic one-body systems on a configurational 2D torus (or 2D lattice) in the Schrödinger picture. The continuum field limit of the former corresponds to the quasiclassical limit of the latter. [S1063-651X(99)15608-3]

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Understanding of long-time dynamics of interacting quantum many-body systems or quantized fields is a long-standing open problem. In particular, one would like to understand the conditions for the emergence of quantum mixing (implying ergodicity and approach to equilibrium in statistical mechanics). Quantum mixing in many-body systems, defined as the decay of general time autocorrelation functions, has recently been studied theoretically [1] and numerically [2,3]. In classical mechanics of a bound Hamiltonian system of a single or few particles, we typically observe transitions from integrable and nonmixing behavior to mixing and generically chaotic behavior [4] when certain interaction parameters are varied, whereas in quantum mechanics we find (see, e.g., [5] for a recent account on the subject) a corresponding transition from integrable behavior, characterized by the existence of quantum numbers, to a statistical behavior, described well by the random matrix theory. Inspired by the rich quantum behavior of nonlinear few-body systems, a few papers have appeared recently concerning quantum chaos in nonlinear many-body systems [6].

However, due to the discreteness of the (quasi)energy spectrum, a finite bound quantum system can never be mixing, whereas a many-body quantum system or a quantized field may acquire a continuous spectrum and become truly mixing in the thermodynamic or continuum limit. Although a generic *strongly* nonintegrable infinite quantum many-body system is expected to be mixing and therefore to obey the standard laws of statistical mechanics, we see no *a priori* reason to exclude nonmixing behavior in the thermodynamic limit of *weakly* nonintegrable many-body systems. Furthermore, numerical evidence has recently been published [3] showing a dynamical phase transition in the thermodynamic limit from nonmixing behavior, exhibiting anomalous-ballistic transport, to mixing behavior, exhibiting normal-diffusive transport, in a family of nonintegrable systems of interacting fermions.

In this paper we wish to propose a nontrivial formal relation between a quasiclassical limit of a vanishing effective Planck constant in a certain nonlinear one-particle system, and a continuum limit of an associated one-dimensional (1D) discrete nonlinear quantum field theory. We do so by constructing explicitly an exact *linear mapping* from a specific large class of infinite interacting spin-$\frac{1}{2}$ chains in the Heisenberg picture to an associated class of nonlinear *one-body-image dynamical systems* (OBI) in the Schrödinger picture which are realized either on a configurational 2D torus or on a 2D lattice [tight-binding (TB) formulation]. By explicitly working out two examples, we will (i) show how integrable behavior of the infinite *XX spin chain in spatially modulated transversal magnetic field* is connected to the integrability of OBI and to the Harper equation [7], and (ii) demonstrate the transition of the related *nonintegrable kicked XX chain* from nonmixing to mixing dynamics corresponding to the stochastic transition from regular to chaotic motion in the classical limit of OBI.

Let $\sigma_j^s, j\in\mathbb{Z}, s\in\{x,y,z\}$ denote an infinite chain of independent Pauli spin-$\frac{1}{2}$ variables. We start by generalizing the result of [8], namely, we find that the operator space spanned by the following set of spatially modulated observables $\{U_n(\vartheta),V_n(\vartheta);n\in\mathbb{Z},\vartheta\in[-\pi,\pi)\}$,

$$U_n(\vartheta) = \sum_{j=-\infty}^{\infty} e^{i[j+(1/2)|n|]\vartheta}\begin{cases}\sigma_j^x\,(\sigma_j^z)_{n-1}\,\sigma_{j+n}^x & n\geq 1,\\ -\sigma_j^z & n=0,\\ \sigma_j^y\,(\sigma_j^z)_{-n-1}\,\sigma_{j-n}^y & n\leq -1,\end{cases}$$

$$V_n(\vartheta) = \sum_{j=-\infty}^{\infty} e^{i[j+(1/2)|n|]\vartheta}\begin{cases}\sigma_j^x\,(\sigma_j^z)_{n-1}\,\sigma_{j+n}^y & n\geq 1,\\ 1 & n=0,\\ -\sigma_j^y\,(\sigma_j^z)_{-n-1}\,\sigma_{j-n}^x & n\leq -1\end{cases}$$

where $(\sigma_j^z)_k := \prod_{l=1}^k \sigma_{j+l}^z$ for $k\geq 1$ and $(\sigma_j^z)_0 := 1$, is closed under the Lie bracket $[A,B]=AB-BA$ and hence forms an infinite-dimensional dynamical Lie algebra (DLA):

$$[U_n(\vartheta),U_l(\varphi)] = 2i\exp\left[\frac{i}{2}(l\vartheta+n\varphi)s_{l-n}\right]V_{n-l}(\vartheta+\varphi) - 2i\exp\left[\frac{i}{2}(l\vartheta+n\varphi)s_{n-l}\right]V_{l-n}(\vartheta+\varphi),$$

$$[U_n(\vartheta),V_l(\varphi)] = 2i\exp\left[\frac{i}{2}(-l\vartheta+n\varphi)s_l\right]U_{n+l}(\vartheta+\varphi) - 2i\exp\left[\frac{i}{2}(l\vartheta+n\varphi)s_l\right]U_{n-l}(\vartheta+\varphi),$$

$$[V_n(\vartheta),V_l(\varphi)]=(s_n+s_l)\left\{\sin\left[\frac{1}{2}(l\vartheta+n\varphi)\right]\times[(s_{n-l}s_l+1)V_{n-l}(\vartheta+\varphi)-(s_{l-n}s_l+1)V_{l-n}(\vartheta+\varphi)]+2\sin\left[\frac{1}{2}(l\vartheta-n\varphi)\right]V_{n+l}(\vartheta+\varphi)\right\}, \tag{1}$$

where $s_n:=-1,0,1$ for $n<,=,>0$, respectively, is a sign of integer n. Checking of the commutation relations (1) is tedious but straightforward work. A few notable members of the DLA are: the Ising or *XX* Hamiltonian, $H_I=JU_1(0)$, $H_{XX}=J[U_1(0)+U_{-1}(0)]$, spin interaction with modulated transversal magnetic field $h_z=h\cos(\epsilon j)$ with a period of $2\pi/\epsilon$ lattice spacings $H_{mh}=\frac{1}{2}h[U_0(\epsilon)+U_0(-\epsilon)]$, spin current $j_s=V_1(0)+V_{-1}(0)$, etc. Let us fix the fundamental modulation wave number ϵ and introduce the following notation:

$$(n,k)\in\mathbb{Z}^2,\qquad U^{\pm}_{n,k}=\frac{1}{2}[U_n(k\epsilon)\pm U_{-n}(-k\epsilon)],$$
$$V^{+}_{n,k}=\frac{1}{2}[V_n(k\epsilon)+V_{-n}(-k\epsilon)]s_n,$$
$$V^{-}_{n,k}=\frac{1}{2}[V_n(k\epsilon)-V_{-n}(-k\epsilon)],$$
$$W^{\pm}_{n,k}=U^{\pm}_{n,k}+iV^{\pm}_{n,k},$$
$$(y,x)\in\mathbb{T}^2,\quad W^{\pm}(y,x)=\frac{1}{2\pi}\sum_{n,k=-\infty}^{\infty}e^{i(ny+kx)}W^{\pm}_{n,k}.$$

We may also consider ϵ as a lattice spacing and treat $W^{\pm}_{n,k}$ as a set of spatially 2π-periodic fields. We will assume that the modulation is *incommensurable* with the lattice spacing, i.e., that $\epsilon/2\pi$ is *irrational*; otherwise, $W^{\pm}_{n,k}$ are periodic with respect to index k. The DLA becomes a Hilbert space when we introduce an infinite temperature (grand) canonical scalar product [8]

$$(A|B):=\lim_{L\to\infty}\frac{1}{L}2^{-L}\mathrm{tr}\,A^{\dagger}B,$$

where L is a diverging length of the spin chain. Let the two linear subspaces of observables spanned by $W^{\sigma}_{n,k}$ [or $W^{\sigma}(y,x)$] for $\sigma\in\{+,-\}$ be denoted by $\mathfrak{M}_{\sigma}$. The spaces $\mathfrak{M}_+$ and $\mathfrak{M}_-$ are *orthogonal* and observables $W^{+}_{n,k}$ and $W^{-}_{n,k}$ (for $n\geq 1$) form an orthonormal basis in each of them, since one can show by direct calculation that

$$(W^{+}_{n,k}|W^{+}_{m,l})=\delta_{n,m}\delta_{k,l},$$
$$(W^{+}_{n,k}|W^{-}_{m,l})=0, \tag{2}$$
$$(W^{-}_{n,k}|W^{-}_{m,l})=(\delta_{n,m}-\delta_{n,-m})\delta_{k,l}.$$

The full set $\{W^{-}_{n,k}\}$ is overcomplete, since $W^{-}_{-n,k}=-W^{-}_{n,k}$, while the subspace $\mathfrak{M}_+=\mathfrak{M}^{\dagger}_+$ is self-adjoint, since $W^{+\dagger}_{n,k}=W^{+}_{-n,-k}$. One can write analogous relations in terms of continuous variables (y,x). We have DLA$=\mathfrak{M}^{\dagger}_-\oplus\mathfrak{M}_+\oplus\mathfrak{M}_-$. Note that the *adjoint map* $(\mathrm{ad}\,A)B=[A,B]$ generates the Heisenberg motion on the DLA, $\exp(it\mathrm{ad}\,A)B=e^{itA}Be^{-itA}$. In particular, the motion generated by $U^{+}_{n,k}$ has a beautiful structure. Let us write the self-adjoint Hamiltonian in a general form as

$$H=\sum_{n,k}\frac{1}{4}h_{n,k}U^{+}_{n,k} \tag{3}$$

using a set of possibly time-dependent complex (Hermitian) coefficients $h_{n,k}=h_{n,k}(t)=h^{*}_{n,-k}$. Tedious but straightforward calculation, using algebra (1), gives the action of ad H on two continuous sets of observables $W^{+}(y,x),W^{-}(y,x)$, $(y,x)\in\mathbb{T}^2$ which can be written in terms of two nonlocal Schrödinger operators $\hat{\mathcal{H}}^{\pm}$:

$$(\mathrm{ad}\,H)W^{\pm}(y,x)=-\frac{1}{\hbar}\hat{\mathcal{H}}^{\pm}W^{\pm}(y,x), \tag{4}$$

$$\hat{\mathcal{H}}^{+}=\sum_{n,k}\hbar g_{n,k}[\sin(n\hat{p}_x-k\hat{p}_y)\sin(kx+ny-\gamma_{n,k})-\sin(n\hat{p}_x+k\hat{p}_y)\sin(kx-ny-\gamma_{n,k})],$$
$$\hat{\mathcal{H}}^{-}=\sum_{n,k}\hbar g_{n,k}[\cos(n\hat{p}_x-k\hat{p}_y)\cos(kx+ny-\gamma_{n,k})+\cos(n\hat{p}_x+k\hat{p}_y)\cos(kx-ny-\gamma_{n,k})], \tag{5}$$

where the coefficients are rewritten in terms of modulus and phase as $h_{n,k}=g_{n,k}\exp(i\gamma_{n,k})$, and $\hat{p}_{x,y}=-i\hbar\,\partial/\partial_{x,y}$ are momentum operators conjugate to x,y with an effective Planck constant [9]

$$\hbar=\frac{1}{2}\epsilon. \tag{6}$$

Since Heisenberg dynamics generated by H are closed on $\mathfrak{M}_{\sigma}$, $(\mathrm{ad}\,H)\mathfrak{M}_{\sigma}\subseteq\mathfrak{M}_{\sigma}$, one may write a general time-evolving operator $A(t)\in\mathfrak{M}_{\sigma}$ in terms of a complex-valued Schrödinger wave function, in either momentum $\Psi^{A}_{n,k}(t)$ or position $\Psi^{A}(y,x;t)$ representation

$$A(t)=\sum_{n,k}\Psi^{A}_{n,k}(t)^{*}W^{\sigma}_{n,k}=\int_{\mathbb{T}^2}dy\,dx\,\Psi^{A}(y,x;t)^{*}W^{\sigma}(y,x). \tag{7}$$

By means of Eq. (4) and the fact that $\hat{\mathcal{H}}^{\sigma}$ is Hermitian on $L^2(\mathbb{T}^2)$ [which can be checked directly using expressions (5)] one can easily show that the Heisenberg evolution of the observable $A(t)$, $(d/dt)A(t)=i(\mathrm{ad}\,H)A$, is *fully equivalent* to the Schrödinger equation

$$i\hbar\frac{d}{dt}\Psi^{A}(y,x;t)=\hat{\mathcal{H}}^{\sigma}\Psi^{A}(y,x;t) \tag{8}$$

governing the time evolution by OBI Hamiltonian $\hat{\mathcal{H}}^{\sigma}$ of one particle on a torus $\mathbb{T}^2$. The *bilinear* map $(H,A(t))$

$\leftrightarrow(\hat{\mathcal{H}}^{\sigma},\Psi^{A}(y,x;t))$ is a central result of this paper. We make the following concluding overview remarks.

(i) There is nontrivial "classical limit" $\hbar\to 0$ of the OBI, *equivalent* (6) to the continuum field limit of the quantum spin chain model $\epsilon\to 0$, if $\hbar h_{n,l}$ (and not $h_{n,l}$ alone) is kept constant and finite.

(ii) The operators $\hat{\mathcal{H}}^{+}$ and $\hat{\mathcal{H}}^{-}$ (5) commute,

$$[\hat{\mathcal{H}}^{+},\hat{\mathcal{H}}^{-}]\equiv 0, \tag{9}$$

and the Poisson bracket of the corresponding classical counterparts vanishes.

(iii) As a consequence of the previous remark, we find that the OBI (8) (and its classical limit) is *integrable*, $\hat{\mathcal{H}}^{-\sigma}$ being the second integral of motion, provided the original spin-field Hamiltonian H (3) or OBI Hamiltonian $\hat{\mathcal{H}}^{\sigma}$ is autonomous, i.e., $(\partial/\partial t)H\equiv 0$. However, one has a possibility of chaotic motion in the classical limit and the emergence of quantum chaos when the problem is explicitly time dependent, for instance, where the coefficients of the Hamiltonian are periodic functions, $h_{n,k}(t+1)=h_{n,k}(t)$. In such a case, one integrates the evolution over one period of time and defines the unitary Floquet maps $U=\hat{\mathcal{T}}\exp[-\int_0^1 dt H(t)]$, $\hat{\mathcal{U}}^{\sigma}=\hat{\mathcal{T}}\exp[-\int_0^1 dt\hat{\mathcal{H}}^{\sigma}(t)]$.

(iv) Temporal correlation functions of the quantum field problem are mapped [using Eqs. (2)] onto transition amplitudes of the OBI

$$(A(t)|B(t'))=\begin{cases}\langle\Psi^{A}(t)|\Psi^{B}(t')\rangle & \sigma=+,\\ \langle\Psi^{A}(t)|\hat{\mathcal{P}}_y\Psi^{B}(t')\rangle & \sigma=-,\end{cases} \tag{10}$$

where $\hat{\mathcal{P}}_y\Psi(y,x)=\Psi(y,x)-\Psi(-y,x)$. Therefore, ergodic properties of many-body dynamics in the DLA are determined by the spectral properties of the OBI: (a) the spin chain is *quantum mixing* in $\mathfrak{M}_{\sigma}$, namely,

$$\lim_{t\to\infty}(A(t)|B)=0,$$

for any $A,B\in\mathfrak{M}_{\sigma}$, if the spectrum of the OBI Hamiltonian $\hat{\mathcal{H}}^{\sigma}$ (or of the OBI Floquet map $\hat{\mathcal{U}}^{\sigma}$) does not have a (nontrivial) *point* component. (b) the spin chain is *quantum ergodic* in $\mathfrak{M}_{\sigma}$, namely,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T dt(A(t)|B)=0,$$

if 0 (or 1) is not in the (nontrivial) point spectrum of $\hat{\mathcal{H}}^{\sigma}$ (or U^{σ}). In the autonomous case, $\partial H/\partial t\equiv 0$, the Hamiltonian H and the *trivial* zero-frequency eigenstate, $\hat{\mathcal{H}}^{+}\Psi^{H}=\Psi^{[H,H]}=0$, $\Psi^{H}_{n,k}=\frac{1}{8}(h_{n,-k}+h_{-n,-k})$ should be excluded from $\mathfrak{M}_{+}$ and $L^2(\mathbb{T}^2)$, respectively. This means that we should then consider above only such pairs of observables $A,B\in\mathfrak{M}_{+}$ which are orthogonal to the Hamiltonian, $(A|H)=(B|H)=0$.

We apply the above results to work out two interesting examples.

Example I. XX spin chain in a spatially modulated quasiperiodic transversal magnetic field $\vec{h}_j=(0,0,h\cos(\epsilon j))$ *(XXmh)*:

$$H=H_{XX}+H_{mh}=JU^{+}_{1,0}+\frac{1}{2}h(U^{+}_{0,1}+U^{+}_{0,-1}). \tag{11}$$

Here the Heisenberg dynamics on DLA is governed by the following commuting one-body problems:

$$\hat{\mathcal{H}}^{+}=\alpha\sin\hat{p}_x\sin y-\beta\sin\hat{p}_y\sin x, \tag{12}$$

$$\hat{\mathcal{H}}^{-}=\alpha\cos\hat{p}_x\cos y+\beta\cos\hat{p}_y\cos x,$$

where $\alpha=2\epsilon J=4\hbar J,\beta=2\epsilon h=4\hbar h$. This model is directly related to the electron motion on a 2D rectangular $a\times b$ lattice in a uniform perpendicular magnetic field h' within the TB approximation [7,10]. In the symmetric gauge $\vec{A}=\frac{1}{2}h'(-y,x,0)$ the TB problem with the band energy $\mathcal{E}(\vec{K})=(\alpha/2)\cos(aK_x)+(\beta/2)\cos(bK_y)$ reads

$$\hat{\mathcal{H}}\Psi_{n,k}=\frac{\alpha}{2}(e^{i(1/2)\epsilon k}\Psi_{n+1,k}+e^{-i(1/2)\epsilon k}\Psi_{n-1,k})+\frac{\beta}{2}(e^{-i(1/2)\epsilon n}\Psi_{n,k+1}+e^{i(1/2)\epsilon n}\Psi_{n,k-1}), \tag{13}$$

where $\epsilon=e_o abh'/c_o\hbar_{\rm phys}$ is here the dimensionless magnetic flux through one lattice cell. We note that discrete indices $(n,k)\in\mathbb{Z}^2$ now label the position lattice (na,kb) while continuous indices $(y,x)\in\mathbb{T}^2$ are the conjugate quasimomenta. OBI Hamiltonians $\hat{\mathcal{H}}^{\pm}$ can be written in terms of $\hat{\mathcal{H}}$ and its *time reversal* $\hat{\mathcal{H}}^{*}=\hat{\mathcal{H}}|_{h'\to -h'}$, namely $\hat{\mathcal{H}}^{\pm}=\hat{\mathcal{H}}\mp\hat{\mathcal{H}}^{*}$, and hence $[\hat{\mathcal{H}},\hat{\mathcal{H}}^{*}]=0$. Using a different, Landau gauge $\vec{A}=h'(0,x,0)$ the TB problem (13) can be rewritten in terms of the 1D Harper equation [7]

$$\frac{1}{2}\alpha(u_{n+1}+u_{n-1})+\beta\cos(n\epsilon-\vartheta)u_n=Eu_n. \tag{14}$$

Let us assume for the moment that $\alpha<\beta$. Then $u_n(\vartheta;E)=u_n$ is a unique *exponentially localized* eigenfunction (EF) of Eq. (14) that has a dense pure point spectrum, and

$$\Psi_{n,k}(\vartheta;E)=\exp\left[i\left(\vartheta-\frac{1}{2}\epsilon n\right)k\right]u_n(\vartheta;E)$$

is a *degenerate dense* set of EFs of the full TB problem (13),

$$\hat{\mathcal{H}}\Psi_{n,k}(\vartheta;E)=E\Psi_{n,k}(\vartheta;E),$$

for a *dense* set of parameters ϑ [10]. Though $\hat{\mathcal{H}}$ and $\hat{\mathcal{H}}^{*}$ should have a common set of EFs, $\Psi_{n,k}(\vartheta;E)$ is not an EF of $\hat{\mathcal{H}}^{*}$, nor is it in L^2, since it is *extended* in the second variable k. We search for a localized EF in both variables with an ansatz

$$\Phi_{n,k}(\vartheta;E,E')=\sum_j v_j\Psi_{n,k}(\vartheta+\epsilon j;E)$$

and require

$$\hat{\mathcal{H}}^{*}\Phi_{n,k}(\vartheta;E,E')=E'\Phi_{n,k}(\vartheta;E,E'),$$

yielding again the Harper equation (14) for the coefficients $v_n=u_n(\vartheta;E')$. In this way we obtain a *complete* common set of EFs of $\hat{\mathcal{H}}$ and $\hat{\mathcal{H}}^*$ in terms of a convolution of two 1D Harper functions multiplied by the appropriate phase factor

$$\Phi_{n,k}(\vartheta;E,E')=\sum_{l}u_{l+n}(\vartheta;E)u_l(\vartheta;E')e^{i[\vartheta-\epsilon l-(1/2)\epsilon n]k}, \tag{15}$$

which is also a common set of EFs of $\hat{\mathcal{H}}^{\pm}$ (12),

$$\hat{\mathcal{H}}^{\pm}\Phi_{n,k}(\vartheta;E,E')=(E\mp E')\Phi_{n,k}(\vartheta;E,E'). \tag{16}$$

The obvious property $\Phi_{n,k}(\vartheta+\epsilon;E,E')=\Phi_{n,k}(\vartheta;E,E')$ suggests the independence of the EFs from parameter ϑ provided $\epsilon/2\pi$ is irrational. If $\alpha>\beta$, localized EFs can be constructed analogously by the duality transformation $n\leftrightarrow k,y\leftrightarrow x$.

Thus we have found that the OBI $\hat{\mathcal{H}}^{\pm}$ have a *dense pure point spectrum* for $\alpha\neq\beta$, hence *XXmh* is nonmixing, nonergodic, and even *completely integrable* system, namely, the zero-energy eigenstates of the Hamiltonians $\hat{\mathcal{H}}^{\sigma}$ are the preimages [under the mapping (7)] of the two infinite sets of *conserved charges*,

$$Q_{\sigma}(E)=\sum_{n,k}\Phi^{*}_{n,k}(\vartheta;E,\sigma E)W^{\sigma}_{n,k},\quad [H,Q_{\sigma}(E)]\equiv 0.$$

The interesting critical case $\alpha=\beta$ where one finds a singular continuous fractal spectrum is not resolved by our analysis.

However, the time correlation function of a certain observable $(A(t)|A)$ may still decay to zero as $t\to\infty$, provided that the observable A is orthogonal to the whole set of conserved charges $Q_{\sigma}(E)$ [11]. Let us assume that $A\in\mathfrak{M}_+$, for simplicity, and put $\vartheta:=0$ for convenience. If the continuous set of orthogonal localized EFs is normalized as localized functions should be, namely, as $\langle\Phi(E,E')|\Phi(E,E')\rangle=1$, then the time-averaged autocorrelation function can be expressed as

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T dt(A(t)|A)=\int dE|\langle\Psi^A|\Phi(E,E)\rangle|^2. \tag{17}$$

Therefore, the averaged correlation function vanishes and observable A behaves as in an ergodic system, if Ψ^A is orthogonal to all diagonal eigenfunctions $\Phi(E,E)$. Let us now assume again that $\alpha<\beta$. Then, the EFs are given by expression (15) from which one easily recognizes the symmetry relation

$$\Phi_{n,0}(E,E)=\Phi_{-n,0}(E,E), \tag{18}$$

from which it follows that $\langle\Psi|\Phi(E,E)\rangle\equiv 0$ for any function of the form $\Psi_{n,k}=\delta_{k,0}(\psi_n-\psi_{-n})$. Interestingly, this is precisely the case for the *spin current* $j_s=W^+_{1,0}-W^+_{-1,0}$, $\Psi^{j_s}_{n,k}=\delta_{k,0}(\delta_{n,1}-\delta_{n,-1})$. Therefore, we have shown that the high-temperature ($\beta\ll 1$) *spin stiffness* $D_s:=\lim_{T\to\infty}\beta/T\int_0^T dt[j_s(t)|j_s]$ is zero, $D_s=0$, and the spin transport is *nonballistic*, if $J<h$ ($\alpha<\beta$), whereas for $J>h$ ($\alpha>\beta$) we find, in general, *ballistic transport*, $D_s>0$, since no such symmetry as Eq. (18) exists if $\alpha>\beta$. We have no further analytical arguments to exclude possible nondecaying oscillations of correlation function $(j_s(t)|j_s)$ in the nonballistic regime, which are, in principle, still possible even though the time-averaged value is zero. In order to clarify this issue, we have performed numerical simulations [which have been performed using the same method as described in Example II (see below), except that the kick parameters α,β have been chosen here to be very small ($\sim 10^{-2}$)] that strongly suggest that in the nonballistic regime ($\alpha<\beta$) the current time correlation function indeed decays as $t\to\infty$, perhaps asymptotically as $(j_s(t)|j_s)\sim 1/t^2$.

Note that the above theoretical result of the transition from ballistic to nonballistic transport can be directly applied to the particle or electrical transport in a 1D chain of free spinless fermions with hopping amplitude $t=J$ in oscillatory chemical potential with amplitude h, since one may use the well known Wigner-Jordan transformation to go from spin to fermion variables.

Example II. Kicked *XXmh* model *(kXXmh)* with periodically time-dependent Hamiltonian

$$H(t)=JU^+_{1,0}+\frac{1}{2}h(U^+_{0,1}+U^+_{0,-1})\sum_m\delta(t-m). \tag{19}$$

The one-period propagator from just after the kick

$$U=\exp\left[-i\frac{1}{2}h(U^+_{0,1}+U^+_{0,-1})\right]\exp(-iJU^+_{1,0}) \tag{20}$$

is equivalent to the Floquet quantum maps of two kicked OBIs:

$$\hat{\mathcal{U}}^{+}=\exp\left(\frac{i\beta}{\hbar}\sin\hat{p}_y\sin x\right)\exp\left(\frac{-i\alpha}{\hbar}\sin\hat{p}_x\sin y\right),$$
$$\hat{\mathcal{U}}^{-}=\exp\left(\frac{-i\beta}{\hbar}\cos\hat{p}_y\cos x\right)\exp\left(\frac{-i\alpha}{\hbar}\cos\hat{p}_x\cos y\right). \tag{21}$$

In the following, we will consider only the map $\hat{\mathcal{U}}^{+}$, since the space $\mathfrak{M}_+$ contains physically more interesting observables, e.g., the spin current. The Floquet evolution $\Psi^A(m)=\hat{\mathcal{U}}^{+m}\Psi^A(0)$, the image of the Heisenberg evolution of observables, $A(m)\in\mathfrak{M}_+$, is in the classical limit equivalent to a symplectic (volume-preserving) (2×2)D map on $\mathbb{T}^2\times\mathbb{R}^2$,

$$x'=x+\alpha\cos p_x\sin y,\quad p'_y=p_y-\alpha\sin p_x\cos y,$$
$$y'=y-\beta\cos p'_y\sin x',\quad p'_x=p_x+\beta\sin p'_y\cos x', \tag{22}$$

which is nonintegrable and (almost) fully chaotic for sufficiently large values of kick parameters, $\alpha,\beta\gg 1$. An interesting question is now if and when the dynamics of *kXXmh* are quantum mixing and how it corresponds to the dynamics of the classical map (22). This problem has been approached numerically by iterating the one-body Floquet map $\hat{\mathcal{U}}^{+}$ on a finite (truncated) momentum space $(n,k)\in\{-N/2\cdots N/2\}^2$. The position states are henceforth discretized as $x_j=sj,y_j$

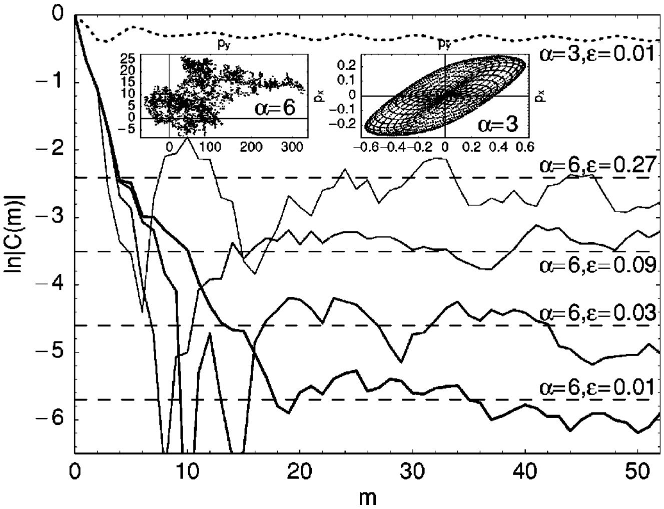

FIG. 1. $\ln|C(m)|$ in *kXXmh* for several indicated values of parameters α and ϵ while $\beta=\alpha/4$. Note the transition to mixing dynamics as ϵ is decreased (from heavy to light full curves) for the chaotic case $\alpha=6$ and stable nonmixing behavior for the quasiregular case $\alpha=3$ [dotted curve, curves for other (small) values of ϵ would be almost indistinguishable]. Broken lines at $\ln|\epsilon/3|$ indicate the scaling $C^*\propto\hbar=\frac{1}{2}\epsilon$. In the two insets we show two orbits in the momentum plane (chaotic, diffusive for $\alpha=6$ and quasiregular for $\alpha=3$) of the map (22) of length 3000 starting at $x_0=0.2, y_0=0, p_{x0}=0, p_{y0}=0.005$.

$=sj$, where $s=2\pi/N$. The truncated Floquet map $\hat{\mathcal{U}}^+$ can be efficiently implemented by means of fast Fourier transformation (FFT), namely, if F is 1D FFT on N sites, then the $N^2\times N^2$ Floquet matrix is decomposed as

$$(F^{-1}\otimes 1)(\operatorname{diag} C_{n,k})(F\otimes F^{-1})(\operatorname{diag} D_{n,k})(1\otimes F),$$

with diagonal matrices $C_{n,k}=\exp[i(\beta/\hbar)\sin sn \sin \hbar k]$ and $D_{n,k}=\exp[-i(\alpha/\hbar)\sin\hbar\, n \sin sk]$. One iteration of the Floquet map requires $\sim 4N^2\log_2 N$ computer operations per time step. In order to avoid recurrences of quantum probability due to finiteness of momentum space, we use an absorbing boundary in momentum space, namely, after each iteration of the truncated Floquet map we multiply the wave function by a box window, $\Psi_{n,k}(m)\to\theta(N/2-\alpha/\hbar-|n|)\,\theta(N/2-\beta/\hbar-|k|)\Psi_{n,k}(m)$. Convergence to the true quantum dynamics on a torus has been checked by comparing results for different truncations, say N and $N/2$, and we went up to $N=2^{14}$.

In Fig. 1 we show numerical results for the autocorrelation function of the spin current $C(m)=(j_s(m)|j_s(0))$ while similar, fully compatible results have been obtained for the time correlations of a few other observables. (i) For sufficiently large kick parameters α,β the classical map (22) is strongly chaotic and mixing, exhibiting normal diffusion in a momentum plane (p_x,p_y). However, *kXXmh* model is not asymptotically (as time $m\to\infty$) mixing for any nonvanishing value of $\hbar$: $|C(m)|$ is rapidly (possibly *exponentially*) decreasing down to some value $C^*=\overline{|C(m)|}$, where it saturates. When we decrease $\hbar$, C^* decreases proportionally, $C^*\propto\hbar$, and hence in the quasiclassical/continuum limit $\hbar=\frac{1}{2}\epsilon\to 0$ the point spectrum of $\hat{\mathcal{U}}^+$ vanishes and *kXXmh* approaches mixing behavior in accordance with the classical map (22). (ii) For smaller but still nonvanishing values of α,β the classical map enters into the regime of Kol'mogorov-Arnold-Moser (KAM) quasiintegrability with invariant tori suppressing the diffusion of momenta (p_x,p_y). Correspondingly, *kXXmh* is strongly nonmixing and $C^*\sim 1$ for any value of $\hbar$. In this regime, $C(m)$ is very weakly $\hbar$ dependent. In both regimes, (i) and (ii), the square widths of the wave packets have been found to be *uniformly* increasing in time, $\langle\Psi^{j_s}(m)|\hat{p}^2_{x,y}|\Psi^{j_s}(m)\rangle\propto m$, and limited only by the size of truncated momentum space N. This rules out the possibility of *quantum localization* and the existence of a *pure point spectrum*, and indicates the coexistence of *point* and *continuous* spectra for any nonvanishing values of $\hbar$ and α,β, a situation which is very similar and possibly related to the 1D kicked Harper model [12]. In the limit $\alpha,\beta\to 0$, the continuous spectral component vanishes and we recover the integrable *XXmh* model with *pure point* spectrum as discussed above. The quantum correlation function $C(m)$ seems to follow the quasiclassical propagator only up to a logarithmically short time, namely, we found empirically that deviation (when it is still small) increases exponentially $|C(m)-C_{\hbar\to 0}(m)|\approx 0.022\hbar^2 e^{\lambda m}$ with $\lambda\approx 0.59$ for $\alpha=3$, $\beta=0.75$ and with $\lambda\approx 1.1$ for $\alpha=6$, $\beta=1.5$.

In a specific infinite-dimensional class of (Pauli spin, or spinless fermion) quantum field models in 1D, the Heisenberg time evolution in two disjoint infinite-dimensional linear subspaces of essential field observables has been shown to be *formally equivalent* to the Schrödinger dynamics of a class of nonlinear one-body image problems on a 2D torus (or 2D lattice). Moreover, the continuum field limit in the original problem has been shown to be equivalent to the (quasi)classical limit in the image problem. Autonomous models of this class were found to be quantum integrable and related to a possibly novel class of integrable classical one-body problems in two dimensions (5, 9). For example, the dynamics of the *XX* chain in a static quasiperiodic transversal field has been solved in terms of the Harper equation [13], and an interesting transition from ballistic to nonballistic spin transport has been discussed. However, time-dependent (e.g., periodically kicked) models of this type behave in a nonintegrable fashion, being mapped onto one-body problems with a chaotic classical limit. It seems that spatial modulation is crucial to breaking integrability since a spin chain kicked with a homogeneous transversal field remains completely integrable [8]. In the continuum field limit our kicked spin chain model (*kXXmh*) has been demonstrated to undergo a (phase) transition from mixing to nonmixing dynamics (similar to a transition found in [3]), as its one-body counterpart in the classical limit undergoes a stochastic transition from chaotic to quasiregular motion. This is an interesting link between quantum field theory and chaotic dynamics and may inspire future research in this direction.

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